

# Random intersection graph process

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## Abstract

We introduce a random intersection graph process aimed at modeling sparse evolving affiliation networks that admit tunable (power law) degree distribution and assortativity and clustering coefficients. We show the asymptotic degree distribution and provide explicit asymptotic formulas for assortativity and clustering coefficients.

**keywords:** random graph process, random intersection graph, degree distribution, power law, clustering, assortativity

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## 1 Introduction

Given non-negative weights  $x = \{x_i\}_{i \geq 1}$  and  $y = \{y_j\}_{j \geq 1}$ , and a nondecreasing positive sequence  $\{\tau(t)\}_{t \geq 1}$ , satisfying  $\lim_{t \rightarrow +\infty} \tau(t) = +\infty$ , let  $H_{x,y}$  be the random bipartite graph with bipartition  $V = \{v_1, v_2, \dots\}$  and  $W = \{w_1, w_2, \dots\}$ , where edges  $\{w_i, v_j\}$  are inserted independently and with probabilities

$$p_{ij} = \min\left\{1, \frac{x_i y_j}{\sqrt{i j}}\right\} \mathbb{I}_{\{a\tau(j) \leq i \leq b\tau(j)\}}. \quad (1)$$

Here  $b > a > 0$  are fixed numbers.  $H_{x,y}$  defines the random intersection graph  $G_{x,y}$  on the vertex set  $V$  such that any  $u, v \in V$  are declared adjacent (denoted  $u \sim v$ ) whenever they have a common neighbor in  $H_{x,y}$ .

Consider, for example, a library where a new item  $w_i$  is acquired at time  $i$ , and where a new user  $v_j$  is registered at time  $j$ . User  $v_j$  picks at random items from a "contemporary literature collection"  $\{w_i : a\tau(j) \leq i \leq b\tau(j)\}$  relevant to time  $j$  (the interval  $\{i : a\tau(j) \leq i \leq b\tau(j)\}$  can also be considered as the lifetime of the user  $v_j$ ). Every actor  $v_j$  and every item  $w_i$  is assigned weight  $y_j$  and  $x_i$  respectively. These weights model the activity of actors and attractiveness of literature items. Now, assume that up to time  $t$  the library has acquired items  $\{w_1, \dots, w_{\tau_*(t)}\} =: W_{\tau_*(t)}$ , where  $\tau_* : \mathbb{N} \rightarrow \mathbb{N}$  is a given nondecreasing function satisfying  $\lim_{t \rightarrow +\infty} \tau_*(t) = +\infty$ . The subgraph  $H_{x,y}(t)$  of  $H_{x,y}$  induced by the bipartition  $V_t = \{v_1, v_2, \dots, v_t\}$  and  $W_{\tau_*(t)}$  defines the random intersection graph  $G_{x,y}(t)$  on the vertex set  $V_t$ : vertices  $u, v \in V_t$  are declared adjacent whenever they have a common neighbor in  $H_{x,y}(t)$ . The graph  $H_{x,y}(t)$  represents a snapshot taken at time  $t$  of the "library" records, while the graph  $H_{x,y}$  shows the complete history of the "library". Graphs  $G_{x,y}(t)$  and  $G_{x,y}$  represent adjacency relations (between users) observed up to time  $t$  and during the whole lifetime of the "library", respectively. Assuming, in addition, that  $x$  and  $y$  are realized values of iid sequences  $X = \{X_i\}_{i \geq 1}$  and  $Y = \{Y_j\}_{j \geq 1}$  we obtain the

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random graph  $G_{X,Y}$  and the random graph process  $\{G_{X,Y}(t)\}_{t \geq 1}$ . The parameters of such a network model are the probability distributions of  $X_1, Y_1$ , the functions  $\tau, \tau^*$  and the cut-off  $a < b$ .

Random intersection graph  $G_{X,Y}$  is aimed at modeling sparse evolving affiliation networks that admit a power law degree distribution and non-vanishing *clustering* and assortativity coefficients. We first observe that choosing inhomogeneous weight sequences  $x$  and  $y$  one typically obtains an inhomogeneous degree sequence of the graph  $G_{x,y}$ : vertices with larger weights attract larger numbers of neighbours. Consequently, in the case where the probability distributions of  $X_1$  and  $Y_1$  have heavy tails, we may expect to obtain a heavy tailed (asymptotic) degree distribution in the random graph  $G_{X,Y}$ . Secondly, we observe that if the set  $W(t)$  of items selected by a user  $v_t$  is (stochastically) bounded and the lifetimes of two neighbours of  $v_t$ , say  $v_s$  and  $v_u$ , intersect, then with a non-vanishing probability  $v_s$  and  $v_u$  share an item from  $W(t)$ . Consequently, the conditional probability  $\alpha_{t|su} = \mathbf{P}(v_s \sim v_u | v_s \sim v_t, v_t \sim v_u)$ , called the clustering coefficient, is positive and bounded away from zero. In particular, the underlying bipartite graph structure serves as a clustering mechanism.

Let us compare our model with the model of evolving network considered recently by Britton, Lindholm and Turova (2011) [6], (see also [5], [18] [19]). In their model vertices are prescribed weights, called social indices, and a vertex  $v_t$  with social index  $s_t$  creates new edges at a rate proportional to  $s_t$ . Clearly, both weight sequences  $\{y_t\}_{t \geq 1}$  and  $\{s_t\}_{t \geq 1}$  have the same purpose of modeling inhomogeneity of adjacency relations (hence both models possess a power law asymptotic degree distribution). But the model of Britton, Lindholm and Turova (2011)[6] does not have the clustering property. We remark, that the role of a bipartite structure in understanding/explaining clustering properties of some social networks has been discussed in Newman, Watts, and Strogatz (2002) [14]. Furthermore, empirically observed clustering properties of real affiliation networks have been reproduced with remarkable accuracy by related models of random intersection graphs, see [2], [3].

In the present paper we only consider the graph  $G_{X,Y}$ . We show the asymptotic distribution of the degree  $d(v_t)$  of a vertex  $v_t$  as time  $t \rightarrow +\infty$ . We also obtain explicit asymptotic expressions for clustering coefficients  $\alpha_{t|s,u}$ ,  $\alpha_{s|t,u}$ ,  $\alpha_{u|s,t}$ , for  $s, t, u \rightarrow +\infty$  such that  $s < t < u$ , and for the assortativity coefficient (Pearson's correlation coefficient between degrees of adjacent vertices)

$$r_{s,t} = \frac{\mathbf{E}_{st}d(v_s)d(v_t) - \mathbf{E}_{st}d(v_s)\mathbf{E}_{st}d_v(t)}{\sqrt{\mathbf{Var}_{st}d(v_s)\mathbf{Var}_{st}d(v_t)}}. \quad (2)$$

Here  $\mathbf{E}_{st}$  denotes the conditional expectation given the event  $v_s \sim v_t$  and  $\mathbf{Var}_{st}d(v_s) = \mathbf{E}_{st}d^2(v_s) - (\mathbf{E}_{st}d_v(s))^2$ . We remark that (empirical) clustering and assortativity coefficients are commonly used characteristics of statistical dependence of adjacency relations of real networks.

Our results are stated in Section 2. Proofs are given in Section 3.

## 2 Results

**Degree.** We first present our results on the asymptotic degree distribution in  $G_{X,Y}$ . We obtain a compound probability distribution in the case where  $\tau(t)$  grows linearly in  $t$  (clustering regime). For  $\tau(t)$  growing faster than linearly in  $t$ , we obtain a mixed Poisson asymptotic degree distribution. We denote  $a_k = \mathbf{E}X_1^k$ , and  $b_k = \mathbf{E}Y_1^k$ .

**Theorem 1.** *Let  $b > a > 0$ . Let  $\tau(t) = t$ . Suppose that  $\mathbf{E}X_1^2 < \infty$  and  $\mathbf{E}Y_1 < \infty$ . For  $t \rightarrow +\infty$*

the random variable  $d(v_t)$  converges in distribution to the random variable

$$d_* = \sum_{j=1}^{\Lambda_1} \varkappa_j, \quad (3)$$

where  $\varkappa_1, \varkappa_2, \dots$  are independent and identically distributed random variables independent of the random variable  $\Lambda_1$ . They are distributed as follows. For  $r = 0, 1, 2, \dots$ , we have

$$\mathbf{P}(\varkappa_1 = r) = \frac{r+1}{\mathbf{E}\Lambda_2} \mathbf{P}(\Lambda_2 = r+1) \quad \text{and} \quad \mathbf{P}(\Lambda_i = r) = \mathbf{E} e^{-\lambda_i} \frac{\lambda_i^r}{r!}, \quad i = 1, 2. \quad (4)$$

Here  $\lambda_1 = 2(b^{1/2} - a^{1/2})a_1Y_1$  and  $\lambda_2 = 2(a^{-1/2} - b^{-1/2})b_1X_1$ .

The second moment condition  $\mathbf{E}X_1^2 < \infty$  of Theorem 1 seems to be redundant.

**Theorem 2.** Let  $b > a > 0$  and  $\nu > 1$ . Let  $\tau(t) = t^\nu$ ,  $t = 1, 2, \dots$ . Suppose that  $\mathbf{E}X_1^2 < \infty$  and  $\mathbf{E}Y_1 < \infty$ . For  $t \rightarrow +\infty$  the random variable  $d(v_t)$  converges in distribution to the random variable  $\Lambda_3$  having the probability distribution

$$\mathbf{P}(\Lambda_3 = r) = \mathbf{E} e^{-\lambda_3} \frac{\lambda_3^r}{r!}, \quad r = 0, 1, 2, \dots \quad (5)$$

Here  $\lambda_3 = \gamma a_2 b_1 Y_1$  and  $\gamma = 4\nu(b^{1/2\nu} - a^{1/2\nu})(a^{-1/2\nu} - b^{-1/2\nu})$ .

*Remark 1.* The result of Theorem 2 extends to a more general class of increasing nonnegative functions  $\tau$ . In particular, assuming that

$$\lim_{t \rightarrow +\infty} \frac{t}{\tau(t)} = 0, \quad \sup_{t > 1} \frac{\tau^{-1}(2t)}{\tau^{-1}(t)} < \infty, \quad (6)$$

and that there exists finite limit

$$\gamma^* = \lim_{t \rightarrow +\infty} t^{-1/2} \sum_{a\tau(t) \leq i \leq b\tau(t)} i^{-1} \sum_{j: a\tau(j) \leq i \leq b\tau(j)} j^{-1/2},$$

we obtain the convergence in distribution of  $d(v_t)$  to  $\Lambda_3$  defined by (5) with  $\lambda_3 = \gamma^* a_2 b_1 Y_1$ . Here  $\tau^{-1}$  denotes the inverse of  $\tau$  (i.e.,  $\tau(\tau^{-1}(t)) = t$ ).

*Remark 2.* The function  $\tau(t) = t \ln t$ , which grows slower than any power  $t^\nu$ ,  $\nu > 1$ , satisfies conditions of Remark 1 with  $\gamma^* = 4(a^{-1/2} - b^{-1/2})(b^{1/2} - a^{1/2})$ . Furthermore, the functions  $\tau_1(t) = e^{\ln^2 t}$  and  $\tau_2(t) = e^t$ , that grow faster than any power  $t^\nu$ , satisfy conditions of Remark 1 with  $\gamma^* = 0$ .

**Clustering.** Our next result, Theorem 3, provides explicit asymptotic formulas for clustering coefficients. We note that for  $s < t < u$  the conditional probabilities  $\alpha_{s|tu}$ ,  $\alpha_{t|su}$  and  $\alpha_{u|st}$  are all different and, given  $0 < a < b$ , mainly depend on the ratios  $s/t$ ,  $s/u$  and  $t/u$ . Denote  $p_\Delta := p_\Delta(s, t, u) = \mathbf{P}(v_s \sim v_t, v_s \sim v_u, v_t \sim v_u)$  the probability that  $v_s, v_t, v_u$  make up a triangle.

**Theorem 3.** Let  $b > a > 0$ . Let  $\tau(t) = t$ . Suppose that  $\mathbf{E}X_1^3 < \infty$  and  $\mathbf{E}Y_1^2 < \infty$ . Assume that  $s, t, u \rightarrow +\infty$  so that  $s < t < u$  and  $\lceil au \rceil \leq \lfloor bs \rfloor$ . We have

$$p_\Delta = \frac{a_3 b_1^3}{\sqrt{stu}} \left( \frac{2}{\sqrt{au}} - \frac{2}{\sqrt{bs}} \right) + o(t^{-2}), \quad (7)$$

$$\alpha_{t|su} = \frac{p_\Delta}{p_\Delta + a_2^2 b_1^2 b_2 t^{-1} (su)^{-1/2} \delta_{t|su}} + o(1), \quad (8)$$

$$\alpha_{s|tu} = \frac{p_\Delta}{p_\Delta + a_2^2 b_1^2 b_2 s^{-1} (tu)^{-1/2} \delta_{s|tu}} + o(1), \quad (9)$$

$$\alpha_{u|st} = \frac{p_\Delta}{p_\Delta + a_2^2 b_1^2 b_2 u^{-1} (st)^{-1/2} \delta_{u|st}} + o(1). \quad (10)$$

Here

$$\begin{aligned}\delta_{t|su} &= \ln(u/t) \ln(t/s) + \ln(u/t) \ln(bs/au) + \ln(t/s) \ln(bs/au) + \ln^2(bs/au), \\ \delta_{s|tu} &= \ln(u/t) \ln(bs/au) + \ln^2(bs/au), \\ \delta_{u|st} &= \ln(t/s) \ln(bs/au) + \ln^2(bs/au).\end{aligned}$$

We remark that the condition  $\lceil au \rceil \leq \lfloor bs \rfloor$  of Theorem 3 excludes the trivial case where  $p_\Delta \equiv 0$ . Indeed, for  $s < u$ , the converse inequality  $\lceil au \rceil > \lfloor bs \rfloor$  means that the lifetimes of  $v_s$  and  $v_u$  do not intersect and, therefore, we have  $\mathbf{P}(v_s \sim v_u) \equiv 0$ . In addition, the inequality  $\lceil au \rceil \leq \lfloor bs \rfloor$  implies that positive numbers  $\delta_{t|su}$ ,  $\delta_{s|tu}$ ,  $\delta_{u|st}$  are bounded from above by a constant (only depending on  $a$  and  $b$ ).

**Assortativity.** Let us now consider the sequence of random variables  $\{d(v_t)\}_{t \geq 1}$ . We assume that  $\tau(t) = t$ . From Theorem 1 we know about the possible limiting distributions for  $d(v_t)$ . Moreover, from the fact that  $G_{X,Y}$  is sparse we can conclude that, for any given  $k$ , the random variables  $d(v_t), d(v_{t+1}), \dots, d(v_{t+k})$  are asymptotically independent as  $t \rightarrow +\infty$ . An interesting question is about the statistical dependence between  $d(v_s)$  and  $d(v_t)$  if we know, in addition, that vertices  $v_s$  and  $v_t$  are adjacent in  $G_{X,Y}$ . We assume that  $s < t$  and let  $s, t \rightarrow +\infty$  so that  $bs - at \rightarrow +\infty$ . Note that the latter condition ensures that the shared lifetime of  $v_s$  and  $v_t$  tends to infinity as  $s, t \rightarrow +\infty$ . In this case we obtain that conditional moments

$$\begin{aligned}\mathbf{E}_{st}d(v_s) &= \mathbf{E}_{st}d(v_t) + o(1) = \delta_1 + o(1), \\ \mathbf{E}_{st}d^2(v_s) &= \mathbf{E}_{st}d^2(v_t) + o(1) = \delta_2 + o(1), \\ \mathbf{E}_{st}d(v_s)d(v_t) &= \delta_2 - \Delta + o(1),\end{aligned}\tag{11}$$

are asymptotically constant. Here  $\Delta = h_1^{-1}(2h_3 + 2h_5 + 4(h_6 - h_7))$  and

$$\delta_1 = 1 + h_1^{-1}(h_2 + 2h_3), \quad \delta_2 = 1 + h_1^{-1}(3h_2 + 6h_3 + h_4 + 6h_5 + 4h_6).$$

Furthermore, we denote

$$\begin{aligned}h_1 &= a_2b_1^2, & h_2 &= a_3b_1^3\tilde{\gamma}, & h_3 &= a_2^2b_1^2b_2\tilde{\gamma}(\sqrt{b} - \sqrt{a}), \\ h_4 &= a_4b_1^4\tilde{\gamma}^2, & h_5 &= a_2a_3b_1^3b_2\tilde{\gamma}^2(\sqrt{b} - \sqrt{a}), \\ h_6 &= a_2^3b_1^3b_3\tilde{\gamma}^2(\sqrt{b} - \sqrt{a})^2, & h_7 &= a_2^3b_1^2b_2^2\tilde{\gamma}^2(\sqrt{b} - \sqrt{a})^2,\end{aligned}\tag{12}$$

and  $\tilde{\gamma} = 2(a^{-1/2} - b^{-1/2})$ . A sketch of the derivation of relations (11) is given in Section 3 below. Finally, from (2) and (11) we obtain that the assortativity coefficient

$$r_{st} = 1 - \frac{\Delta}{\delta_2 - \delta_1^2} + o(1)\tag{13}$$

is asymptotically constant.

We note that each vertex of the graph  $G_{X,Y}$  can be identified with the random subset of  $W$ , consisting of items selected by that vertex, and two vertices are adjacent in  $G_{X,Y}$  whenever their subsets intersect. Graphs describing such adjacency relations between members of a *finite* family  $\tilde{V} = \{\tilde{v}_1, \dots, \tilde{v}_n\}$  of random subsets of a given *finite* set  $\tilde{W} = \{w_1, \dots, w_m\}$  are called random intersection graphs, see [13], [15] and [9]. Our graph  $G_{X,Y}$  is, therefore, a random intersection graph evolving in time. One important application of random intersection graphs, defined by random subsets of fixed size, is the model of a secure wireless sensor network that

uses random predistribution of keys introduced in [8]. Another potential application of random intersection graphs is the statistical analysis and modeling of affiliation networks. For example, they are useful in explaining clustering properties of the actor network, see [2], [3]. Finally, we mention that asymptotic degree distribution and clustering properties of random intersection graphs have been studied in [1], [2], [7], [10], [11], [12], [16].

**Concluding remarks.** We have shown that the random graph  $G_{X,Y}$  admits tunable asymptotic degree distribution (including the power law) and clustering and assortativity coefficients. An interesting problem were to study  $G_{X,Y}$  and  $\{G_{X,Y}(t)\}_{t \geq 1}$  in the case where deterministic cut-offs  $a < b$  in (1) are replaced by random cut-offs  $A_j \leq B_j$  (so that the lifetime  $[A_j\tau(j), B_j\tau(j)]$  of an actor  $v_j$  were random). Furthermore, abrupt cut-offs can be replaced by some smooth cut-off functions.

### 3 Proofs

We first introduce some notation. Then we prove Theorems 1, 2, 3. The proof of Remark 1 goes along the lines of the proof of Theorem 2 and is omitted.

Throughout the proof limits are taken as  $t \rightarrow +\infty$ , if not stated otherwise. By  $c$  we denote positive numbers which may only depend on  $a, b$  and  $\tau$ . We remark that  $c$  may attain different values in different places. We say that a sequence of random variables  $\{\zeta_t\}_{t \geq 1}$  converges to zero in probability (denoted  $\zeta_t = o_P(1)$ ) whenever  $\limsup_t \mathbf{P}(|\zeta_t| > \varepsilon) = 0$  for each  $\varepsilon > 0$ . The sequence  $\{\zeta_t\}_{t \geq 1}$  is called stochastically bounded (denoted  $\zeta_t = O_P(1)$ ) whenever for each  $\delta > 0$  there exists  $N_\delta > 0$  such that  $\limsup_t \mathbf{P}(|\zeta_t| > N_\delta) < \delta$ .

Time intervals

$$T_t = \{i : a\tau(t) \leq i \leq b\tau(t)\}, \quad T_i^* = \{j : a\tau(j) \leq i \leq b\tau(j)\} \quad (14)$$

can be interpreted as lifetimes of the actor  $v_t$  and attribute  $w_i$  respectively. Here and below elements of  $V$  are called actors, elements of  $W$  are called attributes. The oldest and youngest actors that may establish a communication link with  $v_t$  are denoted  $v_{t-}$  and  $v_{t+}$ . Here

$$t_- = \min\{j : T_j \cap T_t \neq \emptyset\}, \quad t_+ = \max\{j : T_j \cap T_t \neq \emptyset\}.$$

The event "edge  $\{w_i, v_j\}$  is present in  $H_{X,Y}$ " is denoted  $w_i \rightarrow v_j$ . Introduce random variables

$$\begin{aligned} \mathbb{I}_{ij} &= \mathbb{I}_{\{w_i \rightarrow v_j\}}, & \mathbb{I}_i &= \mathbb{I}_{it}, & u_i &= \sum_{j \in T_i^* \setminus \{t\}} \mathbb{I}_{ij}, & L &= L_t = \sum_{i \in T_t} u_i \mathbb{I}_i, \\ b_k(I) &= \sum_{j \in I} Y_j^k j^{-k/2}, & a_k(I) &= \sum_{i \in I} X_i^k i^{-k/2}, & I &\subset \mathbb{N}, \end{aligned} \quad (15)$$

$$\lambda_{ij} = X_i Y_j / \sqrt{ij}, \quad Q_{XY}(t) = \sum_{i \in T_t} \lambda_{it} \sum_{j \in T_i^* \setminus \{t\}} \lambda_{ij} \min\{1, \lambda_{ij}\}. \quad (16)$$

We remark, that  $u_i$  counts all neighbours of  $w_i$  in  $H_{X,Y}$  belonging to the set  $V \setminus \{v_t\}$ , and  $L_t$  counts all paths of length 2 in  $H_{X,Y}$  starting from  $v_t$ . Introduce events

$$\mathcal{A}_t = \{\lambda_{it} \leq 1, i \in T_t\}, \quad \mathcal{B}_t(\varepsilon) = \{Y_j \leq \varepsilon^2 j, j \in [t_-, t_+] \setminus \{t\}\}, \quad \varepsilon > 0.$$

By  $\tilde{\mathbf{P}}$  and  $\tilde{\mathbf{E}}$  we denote the conditional probability and expectation given  $X, Y$ . The conditional probability and expectation given  $Y$  is denoted  $\mathbf{P}_X$  and  $\mathbf{E}_X$ . By  $\mathbf{P}_t$  and  $\mathbf{E}_t$  we denote the conditional probability and expectation given  $Y_t$ . By  $d_{TV}(\zeta, \xi)$  we denote the total variation

distance between the probability distributions of random variables  $\zeta$  and  $\xi$ . In the case where  $\zeta, \xi$  and  $X, Y$  are defined on the same probability space, we denote by  $\tilde{d}_{TV}(\zeta, \xi)$  the total variation distance between the conditional distributions of  $\zeta$  and  $\xi$  given  $X, Y$ .

In the proof we use the following simple fact. For a uniformly bounded sequence of random variables  $\{\zeta_t\}_{t \geq 1}$  (i.e.,  $\exists$  nonrandom  $h > 0$  such that  $\forall t$   $0 < \zeta_t < h$  almost surely) we have

$$\zeta_t = o_P(1) \quad \Rightarrow \quad \mathbf{E}\zeta_t = o(1). \quad (17)$$

In particular, given a sequence of bivariate random vectors  $\{(\phi_t, \psi_t)\}_{t \geq 1}$ , defined on the same probability space as  $X, Y$ , we have

$$\tilde{d}_{TV}(\phi_t, \psi_t) = o_P(1) \quad \Rightarrow \quad d_{TV}(\phi_t, \psi_t) = o(1). \quad (18)$$

### 3.1 Proof of Theorem 1

Before the proof we collect auxiliary results. For  $\tau(t) := t$  and  $T_t, T_i^*$  defined in (14), we have

$$\begin{aligned} \sum_{i \in T_t} i^{-1/2} &= t^{1/2}\gamma_1 + rt^{-1/2}, & \sum_{j \in T_i^*} j^{-1/2} &= i^{1/2}\gamma_2 + r'i^{-1/2}, \\ \gamma_1 &:= 2(b^{1/2} - a^{1/2}), & \gamma_2 &:= 2(a^{-1/2} - b^{-1/2}), \end{aligned} \quad (19)$$

where  $|r|, |r'| \leq c$ .

**Lemma 1.** *Let  $t \rightarrow +\infty$ . Assume that  $\mathbf{E}X_1^2 < \infty$  and  $\mathbf{E}Y_1 < \infty$ . We have*

$$\forall \varepsilon > 0 \quad \mathbf{P}(\mathcal{B}_t(\varepsilon)) = 1 - o(1), \quad (20)$$

$$t^{-1}b_2([t_-, t_+] \setminus \{t\}) = o_P(1), \quad (21)$$

$$\mathbf{P}(d(v_t) \neq L_t) = o(1), \quad (22)$$

$$\mathbf{P}(\mathcal{A}_t) = 1 - o(1), \quad (23)$$

$$Q_{XY}(t) = o_P(1), \quad \mathbf{E}Q_{XY}(t) = o(1). \quad (24)$$

For any integers  $t > a^{-1}(b + b^{-1})$  and  $i \in T_t$ , and any  $0 < \varepsilon < 1$  we have

$$|\mathbf{E}a_1(T_t) - a_1\gamma_1 t^{1/2}| \leq ca_1 t^{-1/2}, \quad |\mathbf{E}b_1(T_i^* \setminus \{t\}) - b_1\gamma_2 i^{1/2}| \leq cb_1 i^{-1/2}, \quad (25)$$

$$\mathbf{E}|b_1(T_i^* \setminus \{t\}) - b_1\gamma_2 i^{1/2}| \mathbb{I}_{\mathcal{B}_t(\varepsilon)} \leq ci^{1/2}(\varepsilon b_1^{1/2} + \mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^2 t_-\}}) + cb_1 i^{-1/2}, \quad (26)$$

$$\mathbf{E}|a_1(T_t) - a_1\gamma_1 t^{1/2}| \leq ca_2^{1/2}. \quad (27)$$

*Proof of Lemma 1.* Proof of (20). We estimate the probability of the complement event  $\bar{\mathcal{B}}_t(\varepsilon)$  using the union bound and Markov's inequality

$$\mathbf{P}(\bar{\mathcal{B}}_t(\varepsilon)) \leq \sum_{t_- \leq j \leq t_+} \mathbf{P}(Y_i > \varepsilon^2 j) = t_+ \mathbf{P}(Y_1 > \varepsilon^2 t_-) \leq \varepsilon^{-2}(t_+/t_-) \mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^2 t_-\}} = o(1).$$

Here we estimate  $t_+/t_- \leq c$  and invoke the bound  $\mathbf{E}Y_1 \mathbb{I}_{\{Y_1 > s\}} = o(1)$ , for  $s \rightarrow +\infty$ .

Proof of (21). Denote  $\hat{b}_2(t) = t^{-2} \sum_{1 \leq j \leq t} Y_j^2$ . We note that  $\mathbf{E}Y_1 < \infty$  implies  $\hat{b}_2(t) = o_P(1)$ . The latter bound in combination with the simple inequality  $t_+/t_- \leq c$  implies (21).

Proof of (23). Let  $\bar{\mathcal{A}}_t$  denote the complement event to  $\mathcal{A}_t$ . We have, by the union bound and Markov's inequality,

$$\mathbf{P}_t(\bar{\mathcal{A}}_t) \leq \sum_{i \in T_t} \mathbf{P}_t(\lambda_{it} \geq 1) \leq \sum_{i \in T_t} (it)^{-1} Y_t^2 a_2 \leq ca_2 t^{-1} Y_t^2.$$

We obtain the bound  $\mathbf{P}_t(\bar{\mathcal{A}}_1) = o(1)$ , which implies (23), see (17).

Proof of (22). In view of (17) it suffices to show that  $\mathbf{P}_X(d(v_t) \neq L_t) = o_P(1)$ . We note that  $d(v_t) \neq L_t$  if and only if  $S \geq 1$ , where  $S = \sum' \mathbb{I}_{i_1} \mathbb{I}_{i_2} \mathbb{I}_{i_1 j} \mathbb{I}_{i_2 j}$ . Here we denote  $\sum' = \sum_{\{i_1, i_2\} \subset T_t} \sum_{j \in T_{i_1}^* \cap T_{i_2}^*, j \neq t}$ . Observing that

$$\mathbf{E}_X \mathbb{I}_{i_1} \mathbb{I}_{i_2} \mathbb{I}_{i_1 j} \mathbb{I}_{i_2 j} = \mathbf{E}_X p_{i_1 t} p_{i_2 t} p_{i_1 j} p_{i_2 j} \leq a_2^2 Y_t^2 Y_j^2 / (i_1 i_2 t j)$$

we obtain, by Markov's inequality,

$$\mathbf{P}_X(d(v_t) \neq L_t) = \mathbf{P}_X(S \geq 1) \leq \mathbf{E}_X S \leq a_2^2 Y_t^2 t^{-1} \sum' Y_j^2 (i_1 i_2 j)^{-1}. \quad (28)$$

The simple bound  $\sum_{\{i_1, i_2\} \subset T_t} \frac{1}{i_1 i_2} \leq c$  implies  $\sum' Y_j^2 (i_1 i_2 j)^{-1} \leq c b_2([t_-, t_+] \setminus \{t\})$ . Now, by (21) the right-hand side of (28) tends to zero in probability.

Proof of (24). Denote  $\hat{X}_i = \max\{X_i, 1\}$ ,  $\hat{Y}_j = \max\{Y_j, 1\}$ , and let  $\hat{Q}_{XY}(t)$  denote the sum (16), where  $\lambda_{ij}$  is replaced by  $\hat{\lambda}_{ij} = \hat{X}_i \hat{Y}_j / \sqrt{i j}$ . We observe that  $\mathbf{E} X_1^2 < \infty$  and  $\mathbf{E} Y_1 < \infty$  imply

$$a_\varphi := \mathbf{E} \hat{X}_1^2 \varphi(\hat{X}_1) < \infty, \quad b_\varphi := \mathbf{E} \hat{Y}_1 \varphi(\hat{Y}_1) < \infty,$$

for some positive increasing function  $\varphi : [1, +\infty) \rightarrow [0, +\infty)$  satisfying  $\varphi(u) \rightarrow +\infty$  as  $u \rightarrow +\infty$  (clearly,  $\varphi(\cdot)$  depends on the distributions of  $X_1$  and  $Y_1$ ). In addition, we can choose  $\varphi$  satisfying  $\varphi(u) \leq u$  and  $\varphi(su) \leq \varphi(s)\varphi(u)$ , for  $s, u \geq 1$ . From these inequalities one derives the inequality  $\min\{1, \hat{\lambda}_{ij}\} \leq \varphi(\hat{X}_i) \varphi(\hat{Y}_j) / \varphi(\sqrt{i j})$ . The latter inequality implies

$$\hat{Q}_{XY}(t) \leq \hat{Y}_t Q_{XY}^*(t), \quad Q_{XY}^*(t) := \sum_{i \in T_t} \frac{\hat{X}_i^2 \varphi(\hat{X}_i)}{\sqrt{t i}} \sum_{j \in T_i^* \setminus \{t\}} \frac{\hat{Y}_j \varphi(\hat{Y}_j)}{\sqrt{i j}} \frac{1}{\varphi(\sqrt{i j})}$$

Furthermore, for  $i \in T_t$  and  $j \in T_i^*$  we have  $ij \geq [at]t_- =: t_*^2$ , and  $t_* \rightarrow +\infty$  as  $t \rightarrow +\infty$ . Hence

$$\mathbf{E} Q_{XY}^*(t) \leq \frac{1}{\varphi(t_*)} a_\varphi b_\varphi \sum_{i \in T_t} \frac{1}{\sqrt{t i}} \sum_{j \in T_i^*} \frac{1}{\sqrt{i j}} = O\left(\frac{1}{\varphi(t_*)}\right) = o(1).$$

This bound together with the inequalities  $Q_{XY}(t) \leq \hat{Q}_{XY}(t) \leq \hat{Y}_t Q_{XY}^*(t)$  shows (24).

Proof of (25). These inequalities follow from (19).

Proof of (26). Denote  $\Delta = |\tilde{b} - b_1 \gamma_2 i^{1/2}|$ , where  $\tilde{b}$  denotes the sum  $b_1(T_i^* \setminus \{t\})$ , but with  $Y_j$  replaced by  $\tilde{Y}_j = Y_j \mathbb{I}_{\{Y_j \leq \varepsilon^2 j\}}$ ,  $j \in T_i^* \setminus \{t\}$ . We have

$$|b_1(T_i^* \setminus \{t\}) - b_1 \gamma_2 i^{1/2}| \mathbb{I}_{\mathcal{B}_t(\varepsilon)} = \Delta \mathbb{I}_{\mathcal{B}_t(\varepsilon)} \leq \Delta \leq \Delta_1 + \Delta_2 + \Delta_3, \quad (29)$$

where we denote  $\Delta_1 = |\tilde{b} - \mathbf{E} \tilde{b}|$ ,  $\Delta_2 = |\mathbf{E} \tilde{b} - \mathbf{E} b_1(T_i^* \setminus \{t\})|$ ,  $\Delta_3 = |\mathbf{E} b_1(T_i^* \setminus \{t\}) - b_1 \gamma_2 i^{1/2}|$ . Next, we evaluate  $\mathbf{E} \Delta_1$  and  $\Delta_2$ :

$$(\mathbf{E} \Delta_1)^2 \leq \mathbf{E}(\Delta_1^2) \leq \sum_{j \in T_i^* \setminus \{t\}} j^{-1} \mathbf{E} \tilde{Y}_j^2 \leq \varepsilon^2 b_1 |T_i^*|, \quad (30)$$

$$\Delta_2 \leq \sum_{j \in T_i^* \setminus \{t\}} j^{-1/2} \mathbf{E} Y_j \mathbb{I}_{\{Y_j > \varepsilon^2 j\}} \leq \mathbf{E} Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^2 t_-\}} \sum_{j \in T_i^* \setminus \{t\}} j^{-1/2}. \quad (31)$$

In (30) we first apply Cauchy-Schwartz, then use the linearity of variance of an iid sum, and finally apply the inequality  $\mathbf{Var} \tilde{Y}_j \leq \mathbf{E} \tilde{Y}_j^2 \leq j^{-1} \varepsilon^2 \mathbf{E} Y_j$ . Invoking (25), (30), (31) in (29) and using (19) and  $|T_i^*| \leq ci$  we obtain (26).

Proof of (27). We write  $\mathbf{E}|a_1(T_t) - a_1\gamma_1 t^{1/2}| \leq \mathbf{E}\tilde{\Delta}_1 + \tilde{\Delta}_2$ , where  $\tilde{\Delta}_1 := |a_1(T_t) - \mathbf{E}a_1(T_t)|$  and  $\tilde{\Delta}_2 = |\mathbf{E}a_1(T_t) - a_1\gamma_1 t^{1/2}|$ , and invoke the inequalities

$$(\mathbf{E}\tilde{\Delta}_1)^2 \leq \mathbf{E}\tilde{\Delta}_1^2 = \sum_{i \in T_t} j^{-1}(a_2 - a_1^2) \leq ca_2$$

and  $\tilde{\Delta}_2 \leq ca_1 \leq ca_2^{1/2}$ , see (25). □

Inequality (32) below is referred to as LeCam's inequality, see e.g., [17].

**Lemma 2.** *Let  $S = \mathbb{I}_1 + \mathbb{I}_2 + \dots + \mathbb{I}_n$  be the sum of independent random indicators with probabilities  $\mathbf{P}(\mathbb{I}_i = 1) = p_i$ . Let  $\Lambda$  be Poisson random variable with mean  $p_1 + \dots + p_n$ . The total variation distance between the distributions  $P_S$  of  $P_\Lambda$  of  $S$  and  $\Lambda$*

$$d_{TV}(S, \Lambda) := \sup_{A \subset \{0,1,2,\dots\}} |\mathbf{P}(S \in A) - \mathbf{P}(\Lambda \in A)| \leq \sum_i p_i^2. \quad (32)$$

*Proof of Theorem 1.* Before the proof we introduce some notation. Given  $X, Y$ , we generate independent Poisson random variables

$$\eta_i, \quad \xi_{1i}, \quad \xi_{3i}, \quad \xi_{4i}, \quad \Delta_{ri}, \quad i \in T_t, \quad r = 1, 2, 3,$$

with conditional mean values

$$\begin{aligned} \tilde{\mathbf{E}}\eta_i &= \lambda_{it}, & \tilde{\mathbf{E}}\xi_{1i} &= \sum_{j \in T_i^* \setminus \{t\}} p_{ij}, & \tilde{\mathbf{E}}\xi_{3i} &= X_i b_1 \gamma_2, & \tilde{\mathbf{E}}\xi_{4i} &= X_i \bar{b} i^{-1/2}, \\ \tilde{\mathbf{E}}\Delta_{1i} &= \sum_{j \in T_i^* \setminus \{t\}} (\lambda_{ij} - p_{ij}), & \tilde{\mathbf{E}}\Delta_{2i} &= X_i \delta_{2i} i^{-1/2}, & \tilde{\mathbf{E}}\Delta_{3i} &= X_i \delta_{3i} i^{-1/2}. \end{aligned}$$

Here

$$\delta_{2i} = b_1(T_i^* \setminus \{t\}) - \bar{b}, \quad \delta_{3i} = b_1 \gamma_2 i^{1/2} - \bar{b}, \quad \bar{b} = \min\{b_1(T_i^* \setminus \{t\}), b_1 \gamma_2 i^{1/2}\}.$$

Finally, we define  $\xi_{2i} = \xi_{1i} + \Delta_{1i}$ ,  $i \in T_t$  and introduce random variables

$$L_{0t} = \sum_{i \in T_t} \eta_i u_i, \quad L_{rt} = \sum_{i \in T_t} \eta_i \xi_{ri}, \quad r = 1, 2, 3. \quad (33)$$

We assume, in addition, that given  $X, Y$  the families of random variables  $\{\mathbb{I}_i, i \in T_t\}$  and  $\{\xi_{ri}, i \in T_t, r = 1, 2, 3, 4\}$  are conditionally independent, and that  $\{\eta_i, i \in T_t\}$  is conditionally independent of the set of edges of  $H_{X,Y}$  that are not incident to  $v_t$ .

We are ready to start the proof. In view of (22) the random variables  $d(v_t)$  and  $L_t$  have the same asymptotic distribution (if any). We shall prove that  $L_t$  converges in distribution to  $d_*$ . In the proof we approximate  $L_t$  by the random variable  $L_{3t}$ , see (34) and (35) below. Afterwards we show that  $L_{3t}$  converges in distribution to  $d_*$ .

In order to show that  $L_t$  and  $L_{3t}$  have the same asymptotic distribution (if any) we prove the bounds

$$d_{TV}(L_t, L_{0t}) = o(1), \quad d_{TV}(L_{0t}, L_{1t}) = o(1), \quad (34)$$

$$\mathbf{E}|L_{1t} - L_{2t}| = o(1), \quad \tilde{L}_{2t} - \tilde{L}_{3t} = o_P(1). \quad (35)$$



Here  $\tilde{L}_{2t}$  and  $\tilde{L}_{3t}$  are marginals of the random vector  $(\tilde{L}_{2t}, \tilde{L}_{3t})$  constructed in (39) below which has the property that  $\tilde{L}_{2t}$  has the same distribution as  $L_{2t}$  and  $\tilde{L}_{3t}$  has the same distribution as  $L_{3t}$ .

Let us prove the first bound of (34). We shall show below that

$$\tilde{d}_{TV}(L_t, L_{0t})\mathbb{I}_{\mathcal{A}_t} \leq t^{-1}Y_t^2 a_2(T_t). \quad (36)$$

From the inequality  $\mathbf{E}a_2(T_t) = \sum_{i \in T_t} i^{-1}a_2 \leq c a_2$  we conclude that  $a_2(T_t)$  is stochastically bounded. Hence  $t^{-1}Y_t^2 a_2(T_t) = o_P(1)$ . This bound and (36) combined with (23) imply

$$\tilde{d}_{TV}(L_t, L_{0t}) \leq \tilde{d}_{TV}(L_t, L_{0t})\mathbb{I}_{\mathcal{A}_t} + \mathbb{I}_{\bar{\mathcal{A}}_t} = o_P(1).$$

Now the first bound of (34) follows from (18). It remains to prove (36). We denote  $L'_k = \sum_{i=\lfloor at \rfloor}^k \mathbb{I}_i u_i + \sum_{i=k+1}^{\lfloor bt \rfloor} \eta_i u_i$  and write, by the triangle inequality,

$$\tilde{d}_{TV}(L_t, L_{0t}) \leq \sum_{k \in T_t} \tilde{d}_{TV}(L'_{k-1}, L'_k).$$

Then we estimate  $\tilde{d}_{TV}(L'_{k-1}, L'_k) \leq \tilde{d}_{TV}(\eta_k, \mathbb{I}_k) \leq (kt)^{-1}Y_t^2 X_k^2$ . Here the first inequality follows from the properties of the total variation distance. The second inequality follows from Lemma 2 and the fact that on the event  $\mathcal{A}_t$  we have  $p_{kt} = \lambda_{kt}$ .

Let us prove the second bound of (34). In view of (17) it suffices to show that  $\tilde{d}_{TV}(L_{0t}, L_{1t}) = o_P(1)$ . For this purpose we write, by the triangle inequality,

$$\tilde{d}_{TV}(L_{0t}, L_{1t}) \leq \sum_{k \in T_t} \tilde{d}_{TV}(L_{k-1}^*, L_k^*), \quad (37)$$

where  $L_k^* := \sum_{i=\lfloor at \rfloor}^k \eta_i u_i + \sum_{i=k+1}^{\lfloor bt \rfloor} \eta_i \xi_{1i}$ , and estimate

$$\tilde{d}_{TV}(L_{k-1}^*, L_k^*) \leq \tilde{d}_{TV}(\eta_k u_k, \eta_k \xi_{1k}) \leq \tilde{\mathbf{P}}(\eta_k \neq 0) \tilde{d}_{TV}(u_k, \xi_{1k}). \quad (38)$$

Now, invoking the inequalities

$$\tilde{\mathbf{P}}(\eta_k \neq 0) = 1 - e^{-\lambda_{kt}} \leq \lambda_{kt} \quad \text{and} \quad \tilde{d}_{TV}(u_k, \xi_{1k}) \leq \sum_{j \in T_k^* \setminus \{t\}} p_{kj}^2,$$

see (32), we obtain from (37), (38) and (24) that

$$\tilde{d}_{TV}(L_0, L_1) \leq Q_{XY}(t) = o_P(1).$$

Let us prove the first bound of (35). We observe that

$$|L_{2t} - L_{1t}| = L_{2t} - L_{1t} = \sum_{i \in T_t} \eta_i \Delta_{1i}$$

and

$$\tilde{\mathbf{E}} \sum_{i \in T_t} \eta_i \Delta_{1i} = \sum_{i \in T_t} \lambda_{it} \sum_{j \in T_i^* \setminus \{t\}} (\lambda_{ij} - 1) \mathbb{I}_{\{\lambda_{ij} > 1\}} \leq Q_{XY}(t).$$

We obtain  $\mathbf{E}|L_{2t} - L_{1t}| \leq \mathbf{E}Q_{XY}(t) = o(1)$ , see (24).

Let us prove the second bound of (35). We note that the random vector

$$(\tilde{L}_{2t}, \tilde{L}_{3t}), \quad \tilde{L}_{2t} = \sum_{i \in T_t} \eta_i(\xi_{4i} + \Delta_{2i}), \quad \tilde{L}_{3t} = \sum_{i \in T_t} \eta_i(\xi_{4i} + \Delta_{3i}) \quad (39)$$

has the marginal distributions of  $(L_{2t}, L_{3t})$ . In addition, since  $\Delta_{2i}, \Delta_{3i} \geq 0$  and at most one of them is non-zero, we have  $|\Delta_{2i} - \Delta_{3i}| = \Delta_{2i} + \Delta_{3i}$ . Therefore, we can write

$$\tilde{\Delta} := |\tilde{L}_{2t} - \tilde{L}_{3t}| \leq \sum_{i \in T_t} |\eta_i| |\Delta_{2i} - \Delta_{3i}| = \sum_{i \in T_t} \eta_i(\Delta_{2i} + \Delta_{3i}). \quad (40)$$

We remark that given  $X, Y$  the random variable  $\Delta_{2i} + \Delta_{3i}$  has Poisson distribution with (conditional) mean value

$$\tilde{\mathbf{E}}(\Delta_{2i} + \Delta_{3i}) = X_i i^{-1/2} \delta_i, \quad \delta_i := |b_1(T_i^* \setminus \{t\}) - b_1 \gamma_2 i^{1/2}|.$$

Therefore, (40) implies  $\tilde{\mathbf{E}}\tilde{\Delta} \leq t^{-1/2} Y_t \sum_{i \in T_t} X_i^2 i^{-1} \delta_i$ . Next, for  $0 < \varepsilon < 1$ , we write

$$\mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} \tilde{\Delta} \leq \mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} t^{-1/2} Y_t \sum_{i \in T_t} X_i^2 i^{-1} \delta_i = b_1 a_2 t^{-1/2} \sum_{i \in T_t} i^{-1} \mathbf{E} \delta_i \mathbb{I}_{\mathcal{B}_t(\varepsilon)}.$$

Invoking upper bound (26) for  $\mathbf{E} \delta_i \mathbb{I}_{\mathcal{B}_t(\varepsilon)}$  we obtain  $\mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} \tilde{\Delta} \leq c b_1^{3/2} a_2 \varepsilon + o(1)$ . Finally, this bound combined with Markov's inequality and (20) yields

$$\mathbf{P}(\tilde{\Delta} \geq 1) = \mathbf{P}(\{\tilde{\Delta} \geq 1\} \cap \mathcal{B}_t(\varepsilon)) + o(1) \leq \mathbf{E}\mathbb{I}_{\mathcal{B}_t(\varepsilon)} \tilde{\Delta} + o(1) \leq c b_1^{3/2} a_2 \varepsilon + o(1).$$

We conclude that  $\mathbf{P}(\tilde{\Delta} \neq 0) = \mathbf{P}(\tilde{\Delta} \geq 1) = o(1)$ .

Next we prove that  $L_{3t}$  converges in distribution to  $d_*$  defined by (3). Let  $Y_*$  be a random copy of  $Y_1$ , which is independent of  $X, Y$ . Given  $X, Y, Y_*$ , we generate independent Poisson random variables  $\eta_k^*$ ,  $k \in T_t$  with (conditional) mean values  $\mathbf{E}(\eta_k^* | X, Y, Y_*) = \lambda_{k*}$ , where  $\lambda_{k*} = X_k Y_*(kt)^{-1/2}$ . We assume that, given  $X, Y, Y_*$ , the family of random variables  $\{\eta_k^*, k \in T_t\}$  is conditionally independent of  $\{\xi_{3k}, k \in T_t\}$ . Define  $L_t^* = \sum_{k \in T_t} \eta_k^* \xi_{3k}$ . We note that  $L_t^*$  is defined in the same way as  $L_{3t}$  above, but with  $Y_t$  replaced by  $Y_*$ . Let  $d_*$  be defined in the same way as  $d_*$ , but with  $\lambda_1$  replaced by  $\lambda_* = Y_* a_1 \gamma_1$ . Since  $L_{3t}$  has the same distribution as  $L_t^*$ , and  $d_*$  has the same distribution as  $d_*$ , it suffices to show that  $L_t^*$  converges in distribution to  $d_*$ . For this purpose we show the convergence of Fourier-Stieltjes transforms  $\mathbf{E} e^{iz L_t^*} \rightarrow \mathbf{E} e^{iz d_*}$ , for each  $z \in (-\infty, +\infty)$ . Denote  $\Delta^*(z) = e^{iz L_t^*} - e^{iz d_*}$ . We shall show below that, for any real  $z$  and any realized value  $Y_*$  there exists a positive constant  $c^* = c^*(z, Y_*)$  such that for every  $0 < \varepsilon < 0.5$  we have

$$\limsup_t |\mathbf{E}(\Delta^*(z) | Y_*)| < c^* \varepsilon. \quad (41)$$

Clearly, (41) implies  $\mathbf{E}(\Delta^*(z) | Y_*) = o(1)$ . This fact together with the simple inequality  $|\Delta^*(z)| \leq 2$  yields  $\mathbf{E} \Delta^*(z) = o(1)$ , by Lebesgue's dominated convergence theorem. Finally, the identity  $\mathbf{E} \Delta^*(z) = \mathbf{E} e^{iz L_t^*} - \mathbf{E} e^{iz d_*}$  implies  $\mathbf{E} e^{iz L_t^*} \rightarrow \mathbf{E} e^{iz d_*}$ .

We fix  $0 < \varepsilon < 0.5$  and prove (41). Before the proof we introduce some notation. Denote

$$f_{\mathcal{X}}(z) = \mathbf{E} e^{iz \mathcal{X}_1}, \quad \bar{f}_{\mathcal{X}}(z) = \sum_{r \geq 0} e^{iz r} \bar{p}_r, \quad \bar{p}_r = \bar{\lambda}^{-1} \sum_{k \in T_t} \lambda_{k*} \mathbb{I}_{\{\xi_{3k}=r\}}, \quad \bar{\lambda} = \sum_{k \in T_t} \lambda_{k*},$$

$$\delta = (\bar{f}_{\mathcal{X}}(z) - 1) \bar{\lambda} - (f_{\mathcal{X}}(z) - 1) \lambda_*, \quad f(z) = \mathbf{E}_* e^{iz d_*}, \quad \bar{f}(z) = \bar{\mathbf{E}} e^{iz L_t^*}.$$

Here  $\bar{\mathbf{E}}$  denotes the conditional expectation given  $X, Y, Y_*$  and  $\{\xi_{3k}, k \in T_t\}$ . By  $\mathbf{E}_*$  we denote the conditional expectation given  $Y_*$ .

Introduce the event  $\mathcal{D} = \{|a_1(T_t) - \gamma_1 a_1 t^{1/2}| < \varepsilon t^{1/2} \min\{1, \gamma_1 a_1\}\}$  and let  $\bar{\mathcal{D}}$  denote the complement event. Furthermore, select the number  $T > 1/\varepsilon$  such that  $\mathbf{P}(\varkappa_1 \geq T) < \varepsilon$ . By  $c_1^*, c_2^*, \dots$  we denote positive numbers which do not depend on  $t$ .

We observe that, given  $Y_*$ , the conditional distribution of  $d_*$  is the compound Poisson distribution with the characteristic function  $f(z) = e^{\lambda_*(f_*(z)-1)}$ . Similarly, given  $X, Y, Y_*$  and  $\{\xi_{3k}, k \in T_t\}$ , the conditional distribution of  $L_t^*$  is the compound Poisson distribution with the characteristic function  $\bar{f}(z) = e^{\bar{\lambda}(\bar{f}_*(z)-1)}$ . In the proof of (41) we exploit the convergence  $\bar{\lambda} \rightarrow \lambda_*$  and  $\bar{f}_*(z) \rightarrow f_*(z)$ .

Let us prove (41). We write

$$\mathbf{E}_* \Delta^*(z) = I_1 + I_2, \quad I_1 = \mathbf{E}_* \Delta^*(z) \mathbb{I}_{\mathcal{D}}, \quad I_2 = \mathbf{E}_* \Delta^*(z) \mathbb{I}_{\bar{\mathcal{D}}}.$$

Here  $|I_2| \leq 2\mathbf{P}_*(\bar{\mathcal{D}}) = 2\mathbf{P}(\bar{\mathcal{D}}) = o(1)$ . Indeed, the bound  $\mathbf{P}(\bar{\mathcal{D}}) = o(1)$  follows from (27), by Markov's inequality. Next we estimate  $I_1$ . Combining the identity  $\mathbf{E}_* \Delta^*(z) = \mathbf{E}_* f(z)(e^\delta - 1)$  with the inequalities  $|f(z)| \leq 1$  and  $|e^s - 1| \leq |s|e^{|s|}$ , we obtain

$$|I_1| \leq \mathbf{E}_* |\delta| e^{|\delta|} \mathbb{I}_{\mathcal{D}} \leq c_1^* \mathbf{E}_* |\delta| \mathbb{I}_{\mathcal{D}}. \quad (42)$$

Here we estimated  $e^{|\delta|} \leq e^{6\lambda_*} =: c_1^*$  using the inequalities

$$|\delta| \leq 2\bar{\lambda} + 2\lambda_*, \quad \bar{\lambda} = Y_* t^{-1/2} a_1(T_t) \leq 2\lambda_*.$$

We remark that the last inequality holds provided that event  $\mathcal{D}$  occurs.

Finally, we show that  $\mathbf{E}_* |\delta| \mathbb{I}_{\mathcal{D}} \leq (c_2^* + c_3^* \lambda_* + c_4^* \lambda_*) \varepsilon + o(1)$ . To this aim we write

$$\delta = (\bar{f}_*(z) - 1)(\bar{\lambda} - \lambda_*) + (\bar{f}_*(z) - f_*(z))\lambda_*$$

and estimate  $|\delta| \leq 2|\bar{\lambda} - \lambda_*| + \lambda_* |\bar{f}_*(z) - f_*(z)|$ . The inequality, which holds on the event  $\mathcal{D}$ ,  $|\bar{\lambda} - \lambda_*| \leq Y_* \varepsilon$  implies  $\mathbf{E}_* |\bar{\lambda} - \lambda_*| \mathbb{I}_{\mathcal{D}} \leq c_2^* \varepsilon$  with  $c_2^* := Y_*$ . Next we show that

$$\mathbf{E}_* |\bar{f}_*(z) - f_*(z)| \mathbb{I}_{\mathcal{D}} \leq (c_3^* + c_4^*) \varepsilon + o(1).$$

We first split

$$\bar{f}_*(z) - f_*(z) = \sum_{r \geq 0} e^{izr} (\bar{p}_r - p_r) = R_1 - R_2 + R_3,$$

and then estimate separately the terms

$$R_1 = \sum_{r \geq T} e^{izr} \bar{p}_r, \quad R_2 = \sum_{r \geq T} e^{izr} p_r, \quad R_3 = \sum_{0 \leq r < T} e^{izr} (\bar{p}_r - p_r).$$

Here we denote  $p_r = \mathbf{P}(\varkappa_1 = r)$ . The upper bound for  $R_2$  follows by the choice of  $T$

$$|R_2| \leq \sum_{r \geq T} p_r = \mathbf{P}(\varkappa_1 \geq T) < \varepsilon.$$

Next, combining the identities  $\bar{p}_r = (a_1(T_t))^{-1} \sum_{k \in T_t} k^{-1/2} X_k \mathbb{I}_{\{\xi_{3k}=r\}}$  and

$$\sum_{r \geq T} \sum_{k \in T_t} k^{-1/2} X_k \mathbb{I}_{\{\xi_{3k}=r\}} = \sum_{k \in T_t} k^{-1/2} X_k \mathbb{I}_{\{\xi_{3k} \geq T\}}$$

with the inequality  $a_1(T_t) \geq t^{1/2}a_1\gamma_1/2$ , which holds on the event  $\mathcal{D}$ , we obtain

$$|R_1|\mathbb{I}_{\mathcal{D}} \leq \sum_{r \geq T} \bar{p}_r \leq \frac{2}{a_1\gamma_1 t^{1/2}} \sum_{k \in T_t} \frac{X_k}{k^{1/2}} \mathbb{I}_{\{\xi_{3k} \geq T\}} \leq \frac{2}{a_1\gamma_1 t^{1/2}} \sum_{k \in T_t} \frac{X_k \xi_{3k}}{T k^{1/2}}.$$

Now, the identity  $\mathbf{E}_* X_k \xi_{3k} = a_2 b_1 \gamma_2$  implies  $\mathbf{E}_* |R_1|\mathbb{I}_{\mathcal{D}} \leq c_4^* T^{-1} \leq c_4^* \varepsilon$ .

Now we estimate  $R_3$ . We denote  $p'_r = a_1(T_t)(a_1\gamma_1 t^{1/2})^{-1} \bar{p}_r$  and observe that the inequality  $|a_1(T_t)(a_1\gamma_1 t^{1/2})^{-1} - 1| \leq \varepsilon$ , which holds on the event  $\mathcal{D}$ , implies

$$\left| \sum_{0 \leq r \leq T} e^{itr} (\bar{p}_r - p'_r) \right| \mathbb{I}_{\mathcal{D}} \leq \varepsilon \sum_{0 \leq r \leq T} \bar{p}_r \leq \varepsilon.$$

In the last inequality we use the fact that the probabilities  $\{\bar{p}_r\}_{r \geq 0}$  sum up to 1. It follows now that

$$|R_3|\mathbb{I}_{\mathcal{D}} \leq \varepsilon + \sum_{0 \leq r \leq T} |p'_r - p_r|. \quad (43)$$

Next we estimate

$$\mathbf{E}_* |p'_r - p_r| \leq \mathbf{E}_* |p'_r - \mathbf{E}_* p'_r| + |\mathbf{E}_* p'_r - p_r| \quad (44)$$

where, by the Cauchy-Schwartz and the linearity of the variance of an iid sum, we have

$$(\mathbf{E}_* |p'_r - \mathbf{E}_* p'_r|)^2 \leq \mathbf{E}_* |p'_r - \mathbf{E}_* p'_r|^2 \leq (a_1\gamma_1 t^{1/2})^{-2} a_2(T_t) \leq ct^{-1} a_2 a_1^{-2}, \quad (45)$$

$$|p_r - \mathbf{E}_* p'_r| = p_r |1 - (\gamma_1 t^{1/2})^{-1} \sum_{k \in T_t} k^{-1/2}| \leq ct^{-1}. \quad (46)$$

In (45) we first apply the Cauchy-Schwartz inequality, then use the linearity of variance and the simple inequality  $\mathbf{Var} X_k \mathbb{I}_{\{\xi_{3k}=r\}} \leq a_2$ . In (46) we use the identity  $\mathbf{E}_* X_k \mathbb{I}_{\{\xi_{3k}=r\}} = a_1 p_r$  and (19). From (44), (45), (46) we conclude that  $\mathbf{E}_* |p'_r - p_r| = O(t^{-1/2})$ . Now (43) implies

$$\mathbf{E}_* |R_3|\mathbb{I}_{\mathcal{D}} \leq \varepsilon + O(|T|t^{-1/2}) = \varepsilon + o(1).$$

□

### 3.2 Proof of Theorem 2

Here we assume that  $\tau(t) := t^\nu$ . In the proof below we apply the following simple approximations

$$\begin{aligned} \sum_{k \in T_t} k^{-(1-2\nu)/(2\nu)} &= t^{1/2} \gamma'_1 + r t^{(2\nu)^{-1}-1}, & \sum_{j \in T_k^*} j^{-1/2} &= k^{(2\nu)^{-1}} \gamma'_2 + r' k^{-(2\nu)^{-1}}, \\ \gamma'_1 &:= 2\nu(b^{(2\nu)^{-1}} - a^{(2\nu)^{-1}}), & \gamma'_2 &:= 2(a^{-(2\nu)^{-1}} - b^{-(2\nu)^{-1}}), \end{aligned} \quad (47)$$

where  $|r|, |r'| \leq c$ . We also make use of relations (20), (22), (23) and (24), which remain valid in the case where  $\tau(t) = t^\nu$ , and of the inequalities, for  $k \in T_t$ ,

$$|\mathbf{E} b_1(T_k^* \setminus \{t\}) - b_1 \gamma'_2 k^{1/(2\nu)}| \leq c b_1 k^{-1/(2\nu)}, \quad (48)$$

$$\mathbf{E} |b_1(T_k^* \setminus \{t\}) - b_1 \gamma'_2 k^{1/(2\nu)}| \mathbb{I}_{\mathcal{B}_t(\varepsilon)} \leq c k^{1/(2\nu)} (\varepsilon b_1^{1/2} + \mathbf{E} Y_1 \mathbb{I}_{\{Y_1 > \varepsilon^{2t_-}\}}) + c b_1 k^{-1/(2\nu)}. \quad (49)$$

We note that (48) follows from the second identity of (47), and (49) is obtained in the same way as (26) above.

*Proof of Theorem 2.* Before the proof we introduce some notation. Given  $\varepsilon \in (0, 1)$ , denote

$$\zeta = \sum_{k \in T_t} \lambda_{kt} \zeta_k, \quad \zeta_k = \beta_k b_1 X_k \mathbb{I}'_k, \quad \mathbb{I}'_k = \mathbb{I}_{\{\beta_k b_1 X_k < \varepsilon\}}, \quad \beta_k = k^{(1-\nu)/(2\nu)} \gamma'_2.$$

Given  $X, Y$ , we generate independent Poisson random variables  $\eta_k, \hat{\xi}_{3k}, k \in T_t$ , with (conditional) mean values  $\tilde{\mathbf{E}}\eta_k = \lambda_{kt}$ ,  $\tilde{\mathbf{E}}\hat{\xi}_{3k} = \beta_k b_1 X_k$  and independent Bernoulli random variables  $\tilde{\mathbb{I}}_k, k \in T_t$  with success probabilities

$$\tilde{\mathbf{P}}(\tilde{\mathbb{I}}_k = 1) = 1 - \tilde{\mathbf{P}}(\tilde{\mathbb{I}}_k = 0) = \zeta_k.$$

We assume that, given  $X, Y$ , the sequences  $\{\mathbb{I}_k, k \in T_t\}, \{\tilde{\mathbb{I}}_k, k \in T_t\}, \{\eta_k, k \in T_t\}, \{\hat{\xi}_{3k}, k \in T_t\}$  are conditionally independent. Next, we introduce random variables

$$\hat{L}_{3t} = \sum_{k \in T_t} \eta_k \hat{\xi}_{3k}, \quad L_{4t} = \sum_{k \in T_t} \mathbb{I}_k \hat{\xi}_{3k}, \quad L_{5t} = \sum_{k \in T_t} \mathbb{I}_k \mathbb{I}'_k \hat{\xi}_{3k}, \quad L_{6t} = \sum_{k \in T_t} \mathbb{I}_k \tilde{\mathbb{I}}_k.$$

Furthermore, we define the random variable  $L_{7t}$  as follows. We first generate  $X, Y$ . Then, given  $X, Y$ , we generate a Poisson random variable with the conditional mean value  $\zeta$ . The realized value of the Poisson random variable is denoted  $L_{7t}$ . Thus, we have  $\mathbf{P}(L_{7t} = r) = \mathbf{E}e^{-\zeta} \zeta^r / r!$ , for  $r = 0, 1, \dots$ .

Now we are ready to prove Theorem 2. In the first step of the proof we show that random variables  $d(v_t)$  and  $\hat{L}_{3t}$  have the same asymptotic distribution (if any). Here we proceed as in the proof of (34), (35) above and make use of (20), (22), (23), (24), (48), (49). In the second step we show that  $\hat{L}_{3t}$  converges in distribution to  $\Lambda_3$ . For this purpose we prove that

$$d_{TV}(\hat{L}_{3t}, L_{4t}) = o(1), \quad \mathbf{E}(L_{4t} - L_{5t}) = o(1), \quad (50)$$

$$d_{TV}(L_{6t}, L_{7t}) = o(1), \quad \mathbf{E}e^{izL_{7t}} - \mathbf{E}e^{iz\Lambda_3} = o(1), \quad (51)$$

for every  $-\infty < z < +\infty$ , and that there exists  $c > 0$ , depending only on  $a, b, \nu$ , such that for any  $\varepsilon \in (0, 1)$  we have

$$d_{TV}(L_{5t}, L_{6t}) \leq ca_2 b_1^2 \varepsilon. \quad (52)$$

Let us prove (50), (51), (52). The first bound of (50) is obtained in the same way as the first bound of (34). To show the second bound of (50) we write

$$\tilde{\mathbf{E}}(L_{5t} - L_{6t}) = \sum_{k \in T_t} (1 - \mathbb{I}'_k) \tilde{\mathbf{E}}\mathbb{I}_{kt} \tilde{\mathbf{E}}\hat{\xi}_{3k} = Y_t b_1 t^{-1/2} \sum_{k \in T_t} (1 - \mathbb{I}'_k) X_k^2 \beta_k k^{-1/2}$$

and apply the simple inequality

$$\mathbf{E}X_k^2(1 - \mathbb{I}'_k) \leq \mathbf{E}X_{\underline{t}}^2(1 - \mathbb{I}'_{\underline{t}}), \quad k \in T_t. \quad (53)$$

Here we denote  $\underline{t} = \min\{k : k \in T_t\}$ . We obtain

$$\mathbf{E}(L_{4t} - L_{5t}) = \mathbf{E}\tilde{\mathbf{E}}(L_{4t} - L_{5t}) \leq S_t b_1^2 \mathbf{E}X_{\underline{t}}^2(1 - \mathbb{I}'_{\underline{t}}) = o(1).$$

Here we denote  $S_t = t^{-1/2} \sum_{k \in T_t} \beta_k k^{-1/2}$  and use the simple inequality  $S_t \leq c$ . Furthermore, we invoke the bound  $\mathbf{E}X_{\underline{t}}^2(1 - \mathbb{I}'_{\underline{t}}) = o(1)$ , which holds since  $\underline{t} \rightarrow +\infty$  as  $t \rightarrow +\infty$ .

Let us prove (52). Proceeding as in (37), (38) and using the identity  $\tilde{\mathbb{I}}_k = \tilde{\mathbb{I}}_k \mathbb{I}'_k$  we write

$$\tilde{d}_{TV}(L_{5t}, L_{6t}) \leq \sum_{k \in T_t} \mathbb{I}'_k \tilde{\mathbf{P}}(\mathbb{I}_k \neq 0) \tilde{d}_{TV}(\hat{\xi}_{3k}, \tilde{\mathbb{I}}_k).$$

Next, we estimate  $\mathbb{I}'_k \tilde{d}_{TV}(\hat{\xi}_{3k}, \tilde{\mathbb{I}}_k) \leq \zeta_k^2$ , by LeCam's inequality (32), and invoke the inequality  $\tilde{\mathbf{P}}(\mathbb{I}_k \neq 0) \leq \lambda_{kt}$ . We obtain

$$\tilde{d}_{TV}(L_{5t}, L_{6t}) \leq \sum_{k \in T_t} \mathbb{I}'_k \lambda_{kt} \zeta_k^2 \leq \varepsilon \sum_{k \in T_t} \lambda_{kt} \zeta_k.$$

Here we estimated  $\zeta_k^2 \leq \varepsilon \zeta_k$ . Now the inequalities

$$d_{TV}(L_{5t}, L_{6t}) \leq \mathbf{E} \tilde{d}_{TV}(L_{5t}, L_{6t}) \leq \varepsilon \sum_{k \in T_t} \mathbf{E} \lambda_{kt} \zeta_k \leq a_2 b_1^2 S_t \varepsilon$$

and  $S_t \leq c$  imply (52).

Let us prove the first relation of (51). In view of (17) it suffices to show that  $\tilde{d}_{TV}(L_{6t}, L_{7t}) = o_P(1)$ . For this purpose we write

$$\tilde{d}_{TV}(L_{6t}, L_{7t}) \leq \mathbb{I}_{\mathcal{A}_1} \tilde{d}_{TV}(L_{6t}, L_{7t}) + \mathbb{I}_{\bar{\mathcal{A}}_1},$$

where  $\mathbb{I}_{\bar{\mathcal{A}}_1} = o_P(1)$ , see (23), and estimate using LeCam's inequality (32)

$$\mathbb{I}_{\mathcal{A}_1} \tilde{d}_{TV}(L_{6t}, L_{7t}) \leq \mathbb{I}_{\mathcal{A}_1} \sum_{k \in T_t} \tilde{\mathbf{P}}^2(\mathbb{I}_k \tilde{\mathbb{I}}_k = 1) \mathbb{I}'_k \leq Y_t^2 b_1^2 t^{-1} \sum_{k \in T_t} k^{-1} \beta_k^2 X_k^4 = o_P(1)$$

Here we used the simple inequality  $t^{-1} \sum_{k \in T_t} k^{-1} \beta_k^2 X_k^4 \leq ct^{-2\nu} \sum_{k \leq bt^\nu} X_k^4$  and the fact that  $\mathbf{E} X_1^2 < \infty$  implies the bound  $n^{-2} \sum_{k \leq n} X_k^4 = o_P(1)$ , as  $n \rightarrow +\infty$ .

Finally, we show the second relation of (51). We write  $\tilde{\mathbf{E}} e^{izL_{7t}} = e^{\zeta(e^{iz}-1)}$  and use the bound

$$Y_t b_1 a_2 \gamma - \zeta = o_P(1). \quad (54)$$

We note that, for any real  $z$ , the function  $u \rightarrow e^{u(e^{iz}-1)}$  is bounded and uniformly continuous for  $u \geq 0$ . Therefore, (54) implies the convergence

$$\mathbf{E} e^{izL_{7t}} = \mathbf{E} e^{\zeta(e^{iz}-1)} \rightarrow \mathbf{E} e^{Y_t b_1 a_2 \gamma (e^{iz}-1)} = \mathbf{E} e^{iz\Lambda_3}.$$

It remains to prove (54). We note that (53) implies

$$\zeta = Y_t b_1 \gamma_2'' t^{-1/2} \sum_{k \in T_t} X_k^2 k^{(2\nu)^{-1}-1}. \quad (55)$$

Next, we split  $\gamma = \gamma_1' \gamma_2''$  and invoke the expression for  $\gamma_1'$  obtained from (47). We obtain

$$Y_t b_1 a_2 \gamma = Y_t b_1 \gamma_2'' t^{-1/2} \sum_{k \in T_t} k^{-(1-2\nu)/(2\nu)} a_2 + o_P(1). \quad (56)$$

We observe that (54) follows from (55), (56) and the bound

$$R := t^{-1/2} \sum_{k \in T_t} (a_2 - X_k^2) k^{(2\nu)^{-1}-1} = o_P(1). \quad (57)$$

In the proof of (57) we use the standard truncation argument. Let  $\varepsilon > 0$  and let  $\hat{R}$  be defined as  $R$  above, but with  $X_k^2$  replaced by  $\hat{X}_k^2 = X_k^2 \mathbb{I}_{\{X_k^2 < \varepsilon^2 k\}}$  and  $a_2$  replaced by  $\mathbf{E} \hat{X}_k^2$ . We have  $R = \hat{R} + o_P(1)$  and  $\mathbf{P}(\hat{R} > \varepsilon^{1/2}) \leq \varepsilon^{-1} \mathbf{E} \hat{R}^2 \leq c\varepsilon$ . Letting  $\varepsilon \rightarrow 0$  we obtain  $R = o_P(1)$ .  $\square$

### 3.3 Proof of Theorem 3

Before the proof we state an auxiliary lemma.

**Lemma 3.** Denote  $\mathbf{I}_i^x = \mathbb{I}_{\{X_i > i^{1/2}\}}$  and  $\mathbf{I}_j^y = \mathbb{I}_{\{Y_j > j^{1/2}\}}$ . We have

$$\lambda_{ij}(1 - \mathbf{I}_i^x - \mathbf{I}_j^y) \leq \min\{1, \lambda_{ij}\} \leq \lambda_{ij} \quad (58)$$

*Proof of Lemma 3.* The inequality  $\mathbb{I}_{\{\lambda_{ij} > 1\}} \leq \mathbf{I}_i^x + \mathbf{I}_j^y$  implies

$$\lambda_{ij}(1 - \mathbf{I}_i^x - \mathbf{I}_j^y) \leq \lambda_{ij} - (\lambda_{ij} - 1)\mathbb{I}_{\{\lambda_{ij} > 1\}} = \min\{1, \lambda_{ij}\}.$$

□

*Proof of Theorem 3.* The proof of (9), (8), (10) is very much the same. Therefore, we only prove (7) and (8).

Before the proof we introduce some notation. Denote

$$T_{st} = T_s \cap T_t, \quad T_{tu} = T_t \cap T_u, \quad T_{stu} = T_s \cap T_t \cap T_u, \quad T = T_s \cup T_t \cup T_u.$$

An attribute  $w_i$  is called witness of the edge  $v_j \sim v_k$  whenever  $\mathbb{I}_{ij}\mathbb{I}_{ik} = 1$ . In this case we say that witness  $w_i$  realizes the edge  $v_j \sim v_k$ . Let  $\Delta_1 = \{\exists i : \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{iu} = 1\}$  denote the event that all three edges of the triangle  $v_s, v_t, v_u$  are realized by a common witness. Let  $\Delta_2$  denote the event that all three edges are realized by different witnesses,

$$\Delta_2 = \{\exists \text{ distinct } i, j, k \text{ such that } \mathbb{I}_{is}\mathbb{I}_{it} = 1, \mathbb{I}_{js}\mathbb{I}_{ju} = 1, \mathbb{I}_{kt}\mathbb{I}_{ku} = 1\}.$$

Let  $\Delta = \{v_s \sim v_t, v_s \sim v_u, v_t \sim v_u\}$  denote the event that vertices  $v_s, v_t, v_u$  make up a triangle. Introduce events  $\mathcal{H}_t = \{v_s \sim v_t, v_t \sim v_u\}$  and  $\mathcal{K}_t = \{\exists i \neq j : \mathbb{I}_{it}\mathbb{I}_{is}\mathbb{I}_{jt}\mathbb{I}_{ju} = 1\}$ , and random variables

$$\begin{aligned} S &= \sum_{au \leq k \leq bs} \mathbb{I}_{ks}\mathbb{I}_{kt}\mathbb{I}_{ku}, & Q &= \sum_{au \leq i < j \leq bs} \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{iu}\mathbb{I}_{js}\mathbb{I}_{jt}\mathbb{I}_{ju}, \\ S_t &= \sum_{(i,j) \in I} \mathbb{I}_{it}\mathbb{I}_{is}\mathbb{I}_{jt}\mathbb{I}_{ju}, & Q_t &= \sum_{(i,j) \in I} \sum_{(k,r) \in I, (k,r) \neq (i,j)} \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{jt}\mathbb{I}_{ju}\mathbb{I}_{kt}\mathbb{I}_{ks}\mathbb{I}_{rt}\mathbb{I}_{ru}. \end{aligned}$$

Here  $I$  denote the set of all ordered pairs  $(i, j) \in T \times T$  such that  $i \neq j$ . We remark that every  $(i, j)$  indicates a pair  $(w_i, w_j)$  of possible witnesses of edges  $v_s \sim v_t$  and  $v_t \sim v_u$  respectively. We note that for  $0 < s < t < u$  satisfying  $\lceil au \rceil \leq \lfloor bs \rfloor$  the ratios  $t/s, u/t, u/s \in [1, b/a]$ . Hence the variables  $s, t, u \rightarrow +\infty$  are of the same order of magnitude.

Let us prove (7). We observe that  $\Delta_1 \subset \Delta \subset \Delta_1 \cup \Delta_2$ . Hence

$$\mathbf{P}(\Delta_1) \leq \mathbf{P}(\Delta) \leq \mathbf{P}(\Delta_1) + \mathbf{P}(\Delta_2). \quad (59)$$

Next, by inclusion exclusion, we write  $S - Q \leq \mathbb{I}_{\Delta_1} \leq S$  and estimate

$$\mathbf{E}S - \mathbf{E}Q \leq \mathbf{P}(\Delta_1) \leq \mathbf{E}S. \quad (60)$$

Finally, combining (59) and (60) with the relations

$$\mathbf{E}S = \sum_{au \leq k \leq bs} \frac{\mathbf{E}X_k^3 Y_s Y_t Y_u}{k^{3/2} \sqrt{stu}} + o(t^{-2}) = \frac{a_3 b_1^3}{\sqrt{stu}} \left( \frac{2}{\sqrt{au}} - \frac{2}{\sqrt{bs}} \right) + o(t^{-2}), \quad (61)$$

$$\mathbf{E}Q \leq \sum_{au \leq i < j \leq bs} \mathbf{E}\lambda_{is}\lambda_{it}\lambda_{iu}\lambda_{js}\lambda_{jt}\lambda_{ju} \leq \frac{a_3^2 b_2^3}{stu} \sum_{au \leq i < j \leq bs} \frac{1}{i^{3/2} j^{3/2}} = O(t^{-4}),$$

$$\mathbf{P}(\Delta_2) \leq \mathbf{E} \sum_{i,j,k \in T, i \neq j \neq k} \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{js}\mathbb{I}_{ju}\mathbb{I}_{kt}\mathbb{I}_{ku} \leq \frac{a_2^3 b_2^3}{stu} \left( \sum_{i \in T} i^{-1} \right)^3 = O(t^{-3}), \quad (62)$$

we obtain asymptotic expression (7) for  $p_\Delta = \mathbf{P}(\Delta)$ . We note that in the first step of (61) we apply Lemma 3, and in the last step of (62) we use the inequality  $\sum_{i \in T} i^{-1} \leq c$ .

Let us prove (8). We note that (8) follows from (7) and the relation

$$\mathbf{P}(\mathcal{H}_t) = \mathbf{P}(\Delta) + a_2^2 b_1^2 b_2 \frac{1}{t\sqrt{su}} \delta_{t|su} + o(t^{-2}). \quad (63)$$

It remains to show (63). From the identity  $\mathcal{H}_t = \Delta_1 \cup \mathcal{K}_t$  we obtain

$$\mathbf{P}(\mathcal{H}_t) = \mathbf{P}(\Delta_1) + \mathbf{P}(\mathcal{K}_t) - \mathbf{P}(\Delta_1 \cap \mathcal{K}_t). \quad (64)$$

Next, by inclusion exclusion, we write  $S_t - Q_t \leq \mathbb{I}_{\mathcal{K}_t} \leq S_t$ . These inequalities imply

$$\mathbf{E}S_t - \mathbf{E}S_t(1 - \mathbb{I}_{\mathcal{D}_\varepsilon}) - \mathbf{E}Q_t \mathbb{I}_{\mathcal{D}_\varepsilon} \leq \mathbf{E}\mathbb{I}_{\mathcal{K}_t} \mathbb{I}_{\mathcal{D}_\varepsilon} \leq \mathbf{P}(\mathcal{K}_t) \leq \mathbf{E}S_t. \quad (65)$$

Here the event  $\mathcal{D}_\varepsilon = \{Y_t \leq \varepsilon t\}$  and  $\varepsilon \in (0, 1)$  is non-random. In the remaining part of the proof we show that

$$\mathbf{E}S_t = a_2^2 b_1^2 b_2 \frac{1}{t\sqrt{su}} \delta_{t|s,u} + o(t^{-2}), \quad (66)$$

$$\mathbf{P}(\Delta_1 \cap \mathcal{K}_t) = O(t^{-3}), \quad (67)$$

and that there exists  $c^* > 0$  which does not depend on  $s, t, u$  and  $\varepsilon$  such that, for any  $\varepsilon \in (0, 1)$ ,

$$\mathbf{E}Q_t \mathbb{I}_{\mathcal{D}_\varepsilon} \leq c^* \varepsilon t^{-2} + O(t^{-3}), \quad \mathbf{E}S_t(1 - \mathbb{I}_{\mathcal{D}_\varepsilon}) = o(t^{-2}). \quad (68)$$

We observe that (63) follows from (64), (65), (66) and the bounds (67), (68).

Let us prove (66). Since the product  $\bar{p}_{ij} := p_{is}p_{it}p_{jt}p_{ju}$  is non zero whenever  $i \in T_{st}$  and  $j \in T_{tu}$ , we have

$$\mathbf{E}S_t = \mathbf{E} \sum_{(i,j) \in I} \bar{p}_{ij} = \mathbf{E} \sum_{(i,j): i \in T_{st}, j \in T_{tu}, i \neq j} \bar{p}_{ij}. \quad (69)$$

It is convenient to split the set  $\{(i, j) : i \in T_{st}, j \in T_{tu}, i \neq j\} = \mathbb{T}_1 \cup \dots \cup \mathbb{T}_4$  where

$$\begin{aligned} \mathbb{T}_1 &= (T_{st} \setminus T_u) \times T_{tu}, & \mathbb{T}_2 &= T_{stu} \times (T_{tu} \setminus T_s), \\ \mathbb{T}_3 &= \{(i, j) : i, j \in T_{stu}, i < j\}, & \mathbb{T}_4 &= \{(i, j) : i, j \in T_{stu}, j < i\}. \end{aligned}$$

and write sum (69) in the form

$$\tilde{\mathbf{E}}S_t = S_{t1} + \dots + S_{t4}, \quad S_{tk} := \sum_{(i,j) \in \mathbb{T}_k} \bar{p}_{ij}. \quad (70)$$



Now (66) follows from (70) and the relations, for  $1 \leq k \leq 4$ ,

$$\begin{aligned} \mathbf{E}S_{tk} &= \mathbf{E} \sum_{(i,j) \in \mathbb{T}_k} \lambda_{is} \lambda_{it} \lambda_{jt} \lambda_{ju} + o(t^{-2}) = a_2^2 b_1^2 b_2 \frac{1}{t \sqrt{su}} \sum_{(i,j) \in \mathbb{T}_k} \frac{1}{ij} + o(t^{-2}), \\ \sum_{1 \leq k \leq 4} \sum_{(i,j) \in \mathbb{T}_k} \frac{1}{ij} &= \delta_{t|su} + O(t^{-1}). \end{aligned} \quad (71)$$

In the first step of (71) we used Lemma 3.

Let us prove the first bound of (68). We split the collection of vectors  $(i, j, k, r)$

$$\mathbb{Q} = \{(i, j, k, r) \in T^4 \text{ such that } i \neq j, k \neq r \text{ and } (i, j) \neq (k, r)\}$$

into five non intersecting pieces  $\mathbb{Q} = \mathbb{Q}_1 \cup \dots \cup \mathbb{Q}_5$ , where

$$\begin{aligned} \mathbb{Q}_1 &= \{(i, j, k, r) : i = k\} \cap \mathbb{Q}, & \mathbb{Q}_2 &= \{(i, j, k, r) : i = r\} \cap \mathbb{Q}, \\ \mathbb{Q}_3 &= \{(i, j, k, r) : j = k\} \cap \mathbb{Q}, & \mathbb{Q}_4 &= \{(i, j, k, r) : j = r\} \cap \mathbb{Q}, \end{aligned}$$

and  $\mathbb{Q}_5 = \{(i, j, k, r) : \text{all } i, j, k, r \text{ are distinct}\} \cap \mathbb{Q}$ , and write

$$Q_t = \sum_{1 \leq z \leq 5} Q_{tz}, \quad Q_{tz} = \sum_{(i,j,k,r) \in \mathbb{Q}_z} \mathbb{I}_{is} \mathbb{I}_{it} \mathbb{I}_{jt} \mathbb{I}_{ju} \mathbb{I}_{ks} \mathbb{I}_{kt} \mathbb{I}_{rt} \mathbb{I}_{ru}.$$

Denote  $\tilde{\mathbb{Q}} = \{(i, j, r) \in T^3 : \text{all } i, j, r \text{ are distinct}\}$ . Observing that the typical summand of the sum  $Q_{t1}$  is  $\mathbb{I}_{is} \mathbb{I}_{it} \mathbb{I}_{jt} \mathbb{I}_{ju} \mathbb{I}_{rt} \mathbb{I}_{ru}$  (since  $i = k$ ), we write

$$\begin{aligned} \mathbf{E}Q_{t1} \mathbb{I}_{\mathcal{D}_\varepsilon} &\leq \mathbf{E} \sum_{(i,j,r) \in \tilde{\mathbb{Q}}} \lambda_{is} \lambda_{it} \lambda_{jt} \lambda_{ju} \lambda_{rt} \lambda_{ru} \mathbb{I}_{\mathcal{D}_\varepsilon} \\ &\leq \frac{a_2^3}{s^{1/2} t^{3/2} u} \mathbf{E} Y_s Y_t^3 Y_u^2 \mathbb{I}_{\mathcal{D}_\varepsilon} \left( \sum_{i \in T} \frac{1}{i} \right)^3 \\ &\leq c^3 \varepsilon \frac{a_2^3}{s^{1/2} t^{1/2} u} \mathbf{E} Y_s Y_t^2 Y_u^2 \\ &\leq c' \varepsilon t^{-2}. \end{aligned}$$

Here used inequalities  $Y_t t^{-1} \mathbb{I}_{\mathcal{D}_\varepsilon} \leq \varepsilon$  and  $\sum_{i \in T} \frac{1}{i} \leq c$ . Similarly, we prove the inequality  $\mathbf{E}Q_{t4} \mathbb{I}_{\mathcal{D}_\varepsilon} \leq c' \varepsilon t^{-2}$ . Furthermore, observing that the typical summand of the sum  $Q_{t2}$  is  $\mathbb{I}_{is} \mathbb{I}_{it} \mathbb{I}_{iu} \mathbb{I}_{jt} \mathbb{I}_{ju} \mathbb{I}_{ks} \mathbb{I}_{kt}$  (since  $i = r$ ), we write

$$\begin{aligned} \mathbf{E}Q_{t2} \mathbb{I}_{\mathcal{D}_\varepsilon} &\leq \mathbf{E} \sum_{(i,j,k) \in \tilde{\mathbb{Q}}} \lambda_{is} \lambda_{it} \lambda_{iu} \lambda_{jt} \lambda_{ju} \lambda_{ks} \lambda_{kt} \mathbb{I}_{\mathcal{D}_\varepsilon} \\ &\leq \frac{a_3 a_2^2}{s t^{3/2} u} \mathbf{E} Y_s^2 Y_t^3 Y_u^2 \mathbb{I}_{\mathcal{D}_\varepsilon} \left( \sum_{i \in T} \frac{1}{i} \right)^2 \left( \sum_{i \in T} \frac{1}{i^{3/2}} \right) \\ &\leq c^3 \frac{a_3 a_2^2}{s t u} \mathbf{E} Y_s^2 Y_t^2 Y_u^2. \end{aligned}$$

In the last step we used inequalities  $Y_t t^{-1} \mathbb{I}_{\mathcal{D}_\varepsilon} \leq 1$  and  $\sum_{i \in T} \frac{1}{i^{3/2}} \leq c t^{-1/2}$ . Hence,  $\mathbf{E}Q_{t2} \mathbb{I}_{\mathcal{D}_\varepsilon} =$

$O(t^{-3})$ . Similarly, we prove the bound  $\mathbf{E}Q_{t3}\mathbb{I}_{\mathcal{D}_\varepsilon} = O(t^{-3})$ . Finally, we estimate

$$\begin{aligned}\mathbf{E}Q_{t5}\mathbb{I}_{\mathcal{D}_\varepsilon} &\leq \mathbf{E} \sum_{(i,j,k,r) \in \mathbb{Q}_5} \lambda_{is}\lambda_{is}\lambda_{jt}\lambda_{ju}\lambda_{ks}\lambda_{kt}\lambda_{rt}\lambda_{ru}\mathbb{I}_{\mathcal{D}_\varepsilon} \\ &\leq \frac{a_2^4}{st^2u} \mathbf{E}Y_s^2Y_t^4Y_u^2\mathbb{I}_{\mathcal{D}_\varepsilon} \left( \sum_{i \in T} \frac{1}{i} \right)^4 \\ &\leq c'\varepsilon^2t^{-2}.\end{aligned}$$

In the last step we used the inequality  $Y_t^2t^{-2}\mathbb{I}_{\mathcal{D}_\varepsilon} \leq \varepsilon^2$ .

Let us prove the second bound of (68). We have

$$\mathbf{E}S_t(1 - \mathbb{I}_{\mathcal{D}_\varepsilon}) \leq \mathbf{E} \sum_{i,j \in T, i \neq j} \lambda_{is}\lambda_{it}\lambda_{jt}\lambda_{ju}(1 - \mathbb{I}_{\mathcal{D}_\varepsilon}) \leq \frac{a_2^2b_1^2}{st} \mathbf{E}Y_t^2\mathbb{I}_{\{Y_t \geq \varepsilon t\}} \left( \sum_{i \in T} i^{-1} \right)^2 = o(t^{-2}).$$

Let us prove (67). The inequalities  $\mathbb{I}_{\mathcal{K}_t} \leq S_t$ ,  $\mathbb{I}_{\Delta_1} \leq S$  and  $S \leq \tilde{S}$ , where

$$\tilde{S} = \sum_{k \in T} \mathbb{I}_k^* \quad \text{and} \quad \mathbb{I}_k^* = \mathbb{I}_{ks}\mathbb{I}_{kt}\mathbb{I}_{ku},$$

imply  $\mathbf{P}(\Delta_1 \cap \mathcal{K}_t) = \mathbf{E}\mathbb{I}_{\Delta_1}\mathbb{I}_{\mathcal{K}_t} \leq \mathbf{E}S_t\tilde{S}$ . We show that  $\mathbf{E}S_t\tilde{S} = O(t^{-3})$ . We split  $S_t\tilde{S} = \tilde{S}_1 + \tilde{S}_2$ ,

$$\tilde{S}_1 = \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{jt}\mathbb{I}_{ju}(\mathbb{I}_i^* + \mathbb{I}_j^*), \quad \tilde{S}_2 = \sum_{(i,j,k) \in \tilde{\mathbb{Q}}} \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{jt}\mathbb{I}_{ju}\mathbb{I}_k^*,$$

and estimate

$$\begin{aligned}\mathbf{E}\tilde{S}_1 &\leq \mathbf{E} \sum_{i \in T} \sum_{j \in T \setminus \{i\}} \lambda_{is}\lambda_{it}\lambda_{jt}\lambda_{ju}(\lambda_{iu} + \lambda_{js}) = O(t^{-3}), \\ \mathbf{E}\tilde{S}_2 &\leq \mathbf{E}\tilde{S}_2' \leq \mathbf{E} \sum_{(i,j,k) \in \tilde{\mathbb{Q}}} \lambda_{is}\lambda_{it}\lambda_{jt}\lambda_{ju}\lambda_{ks}\lambda_{ku} = O(t^{-3}).\end{aligned}\tag{72}$$

Here  $\tilde{S}_2'$  is defined in the same way as  $\tilde{S}_2$ , but with  $\mathbb{I}_k^*$  replaced by  $\mathbb{I}_k' = \mathbb{I}_{ks}\mathbb{I}_{ku}$ .

□

### 3.4 Proof of (11)

We only give a sketch of the proof. Let  $s < t$  satisfy the inequality  $\lceil at \rceil \leq \lfloor bs \rfloor$ . An attribute  $w_i$  is called witness of the edge  $v_s \sim v_t$  whenever  $\mathbb{I}_{it}\mathbb{I}_{is} = 1$ . The sums

$$e_{st} = \sum_{i \in T_s \cap T_t} \mathbb{I}_{is}\mathbb{I}_{it} \quad \text{and} \quad q_{st} = \sum_{\{i,j\} \subset T_s \cap T_t} \mathbb{I}_{is}\mathbb{I}_{it}\mathbb{I}_{js}\mathbb{I}_{jt}$$

count witnesses and pairs of witnesses of the edge  $v_s \sim v_t$ , respectively. We write, by inclusion-exclusion,

$$e_{st} - q_{st} \leq \mathbb{I}_{\{v_s \sim v_t\}} \leq e_{st}\tag{73}$$

and note that the quadratic term  $q_{st}$  is negligibly small. Hence, we approximate

$$\mathbb{I}_{\{v_s \sim v_t\}} = e_{st}(1 + o_P(1)), \quad \mathbf{P}(v_s \sim v_t) = (1 + o(1))\mathbf{E}e_{st}.\tag{74}$$

Given  $t$  and  $i, j \in T_t$ , we denote  $T_{it}^* = T_i^* \setminus \{t\}$  and introduce random variables

$$u_{it} = \sum_{k \in T_{it}^*} \mathbb{I}_{ik}, \quad z_{ijt} = \sum_{k \in T_{it}^* \cap T_{jt}^*} \mathbb{I}_{ik} \mathbb{I}_{jk}, \quad L_t = \sum_{i \in T_t} \mathbb{I}_{it} u_{it}, \quad Q_t = \sum_{\{i,j\} \subset T_t} \mathbb{I}_{it} \mathbb{I}_{jt} z_{ijt}.$$

We remark that  $L_t$  counts pairs  $(v_s \sim v_t; w_i)$ , where  $w_i$  is a witness of the edge  $v_s \sim v_t$  in  $G_{X,Y}$ , for some  $v_s \in W \setminus \{v_t\}$ . In particular, we have  $d(v_t) \leq L_t$ . Similarly,  $Q_t$  counts all triples  $(v_s \sim v_t; w_i, w_j)$ , where  $w_i$  and  $w_j$  are distinct witnesses of an edge  $v_s \sim v_t$ . Note that a neighbour  $v_s$  of  $v_t$ , which has  $k$  witnesses of the edge  $\{v_s \sim v_t\}$ , contributes 1 to the number  $d(v_t)$  of neighbours of  $v_t$ . It contributes  $k$  to the sum  $L_t$  and it contributes  $\binom{k}{2}$  to the sum  $Q_t$ . Hence, we always have

$$L_t - Q_t \leq d(v_t) \leq L_t.$$

We note that the quadratic term  $Q_t$  is negligibly small and approximate  $d(v_t) = L_t(1 + o_P(1))$ . Combining this approximation with (74) we obtain, for  $r = 1, 2$  and  $u = s, t$ ,

$$\mathbf{E}_{st} d^r(v_u) = (\mathbf{E}_{est})^{-1} \mathbf{E}_{est} L_u^r + o(1) \quad \text{and} \quad \mathbf{E}_{st} d(v_s) d(v_t) = (\mathbf{E}_{est})^{-1} \mathbf{E}_{est} L_s L_t + o(1). \quad (75)$$

Next we evaluate expectations in the right-hand sides of (75). A straightforward but tedious calculation shows that

$$\begin{aligned} \mathbf{E}_{est} &= \Theta(1 + o(1)) h_1, \\ \mathbf{E}_{est} L_s &= \mathbf{E}_{est} L_t + o(\Theta) = \Theta(1 + o(1)) (h_1 + h_2 + 2h_3), \\ \mathbf{E}_{est} L_s^2 &= \mathbf{E}_{est} L_t^2 + o(\Theta) = \Theta(1 + o(1)) (h_1 + 3h_2 + 6h_3 + h_4 + 6h_5 + 4h_6), \\ \mathbf{E}_{est} L_s L_t &= \Theta(1 + o(1)) (h_1 + 3h_2 + 4h_3 + h_4 + 4h_5 + 4h_7). \end{aligned}$$

Here we denote  $\Theta = (st)^{-1/2} \ln(bs/at)$ . We recall that  $h_i$  are defined in (12) above. Now (11) follows from (75).

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